JOURNAL OF STATISTICAL MODELING & ANALYTICS (JOSMA)
(ISSN: 2180-3102)
UNIVERSITI MALAYA

On the Type I Half-Logistic Exponentiated Lomax Distribution with Applications

Sule Omeiza Bashiru[1*], Ibrahim Ismail Itopa[2] and Alhaji Modu Isa[3]

[1]*Department of Mathematics and Statistics, Confluence University of Science and Technology, Osara, Kogi State, Nigeria*
[2]*Department of Mathematical Sciences, Prince Abubakar Audu University, Anyigba, Kogi State, Nigeria*
[3]*Department of Mathematics and Computer Science, Borno State University, Maiduguri, Nigeria*

**Corresponding Author: bash0140@gmail.com*

RESEARCH ARTICLE

Received 21 July 2023
Accepted 10 May 2024
Published 30 May 2024

Abstract

In this research, a new four-parameter lifetime distribution called the Type I Half-Logistic exponentiated Lomax distribution is introduced. The study presents various mathematical properties of this novel model, including moments, moment generating function, quantile function, survival function, and hazard function. Additionally, the behavior of the probability density function for both the maximum and minimum order statistics is analyzed within the framework of this new distribution. The paper also explores parameter estimation using the maximum likelihood method to estimate the distribution's unknown parameters. To validate the practicality and versatility of the proposed distribution, three real-life datasets are used for empirical validation. Comparisons are made with existing lifetime distributions from the literature, and the results consistently demonstrate that the proposed distribution outperforms others in terms of fit and flexibility for all three datasets. This highlights its effectiveness in modeling diverse real-world scenarios. The introduction of the Type I Half-Logistic exponentiated Lomax distribution contributes to the expansion of distribution theory and offers a valuable tool for researchers, statisticians, and analysts to accurately model and analyze lifetime data in various fields. The analytical properties derived in this study provide a solid foundation for future research and applications, and the successful parameter estimation and goodness-of-fit results further validate the practical significance of the proposed distribution.

Keywords: Bladder cancer patients, Flexibility, Half-logistic exponentiated, Information criteria, Lomax distribution.

1. Introduction

In the last decade, the field of distribution theory has witnessed a surge in research focused on developing novel continuous probability distributions with wide-ranging applications in diverse fields such as engineering, medicine, insurance, and more. This growing interest can be attributed to the limitations of many existing distributions in adequately modeling datasets with heavy tails and effectively controlling both skewness and kurtosis, particularly in scenarios where the distribution is characterized solely by a scale parameter, as exemplified by the exponential and inverse exponential distributions. Consequently, several studies have been undertaken to extend or generalize existing distributions, introducing greater flexibility to classical distributions through the introduction of one or more parameters, giving rise to a plethora of families of distributions proposed in the literature.

Numerous families of distributions have emerged from the endeavors of researchers in the field of distribution theory, significantly enriching the repertoire of available statistical tools. Some notable works contributing to these families include Bello et al. (2021a), Bello et al. (2021b), Ibrahim et al. (2020a), Ibrahim et al. (2020b), Elgarhy et al. (2017), Elgarhy et al. (2018), Cordeiro and deCastro (2011), and Falgore and Doguwa (2020). These families of distributions have proven instrumental in addressing the limitations of traditional distributions and have expanded their utility across diverse fields. Researchers and practitioners have been empowered to achieve enhanced accuracy and precision in modeling and analyzing real-world data, thanks to the flexibility and adaptability introduced by these families of distributions. Furthermore, the advancements in distribution theory have fostered a deeper understanding of complex data patterns, heavy-tailed datasets, and skewed distributions. The exploration of various distribution families has significantly improved statistical analysis in diverse fields such as engineering, medicine, finance, and more. These distributions have inspired further research, leading to an ever-expanding list of models tailored to specific data characteristics. As distribution theory advances, the synergy between researchers from different disciplines fosters innovative solutions and supports data-driven decision-making. Ultimately, the use of these distribution families drives innovation and benefits society as a whole.

A popular method employed to extend existing distributions is by fitting a baseline distribution into a generalized family of distributions. This approach aims to enhance the baseline distribution's flexibility and adaptability to different datasets with diverse characteristics. By integrating the baseline distribution into a family of distributions, additional parameters are introduced, thereby empowering the baseline distribution to better fit a wide range of data patterns. This hybrid distribution can then be used to model real-world data with improved accuracy and robustness. The baseline statistical distribution under consideration in this research is the Lomax Distribution, also known as the Pareto distribution of the second kind. Since its introduction by Lomax (1954), this distribution has gained substantial prominence and found widespread use in various fields. Originally employed for fitting business failure data in life testing, its utility has expanded to encompass broader realms, including reliability and life testing, as demonstrated by the research of Hassan and Al-Ghamdi (2009). Moreover, the Lomax distribution has demonstrated its adaptability and practicality across a diverse spectrum of disciplines. In the biological sciences, it has been effectively utilized as a modeling tool for various phenomena (Tahir et al., 2015). Meanwhile, in computer science, researchers have successfully applied it to describe the distribution of sizes of computer files on servers, as evidenced by the comprehensive studies conducted by Holland et al. (2006).

The Lomax distribution is a versatile and widely applicable statistical tool, used to tackle various real-world scenarios. Researchers and practitioners recognize its flexibility and predictive capabilities, contributing to advancements in statistical analysis and insights into complex data distributions. Numerous researchers have extended and modified its applications, leading to a deeper understanding of its potential uses. Named after the Russian mathematician Boris Lomax, this distribution finds applications across diverse fields due to its adaptability to real-world scenarios. Recent influential works include those by Nagarjuna et al. (2022), Sule et al. (2021a and 2021b), Mead (2016), Rady et al. (2016), Hassan and Abd-Allah (2018), and Nagarjuna et al. (2021a and 2021b), which have significantly broadened its applicability. The simplicity and versatility of the Lomax distribution are evident in its two parameters defining its shape and scale. Extensions proposed by researchers have added more parameters or constraints, enhancing its adaptability to fit a wide range of datasets more accurately. The cumulative efforts to refine the Lomax distribution have broadened its scope of application. It is now frequently used in finance, engineering, biology, environmental science, and social sciences. For instance, it models extreme events in finance, analyzes the distribution of extreme weather events in environmental science, and studies the occurrence of rare diseases in biology (Ijaz et al., 2019). Advancements in distribution theory, like the Lomax distribution, lead to a richer toolbox of statistical tools for analyzing complex data and making informed decisions in various industries.

Hence, the Lomax distribution has evolved significantly due to the contributions of various researchers, resulting in a powerful statistical tool with widespread applicability. The extensions and modifications have endowed it with greater flexibility, enabling it to tackle a broader array of real-world scenarios. As distribution theory continues to progress, the Lomax distribution stands as a testament to the value of refining and expanding existing statistical models, promising further advancements in statistical analysis across numerous fields.

The cumulative distribution function (CDF) and probability density function (PDF) of the Lomax distribution are provided below in equations (1) and (2), respectively:

$$G(x;\theta,\beta)=1-\left(1+\beta x\right)^{-\theta}, \tag{1}$$

$$g(x;\theta,\beta)=\theta\beta\left(1+\beta x\right)^{-(\theta+1)}, \tag{2}$$

where $x\geq 0$, $\beta>0$ is the scale parameter and $\theta>0$ is the shape parameter.

The focus of this study lies on a particular generator introduced by Bello et al. (2021a), known as the Type I Half-Logistic Exponentiated family of distribution. By utilizing this generator and substituting the Lomax distribution, a novel distribution emerges, named the Type I Half-logistic exponentiated Lomax (TLExLx) distribution. Bello et al. (2021a) generator incorporates two shape parameters, imparting increased skewness to the baseline distribution and bolstering its capability to accurately fit data sets with diverse degrees of skewness. The Type I Half-Logistic Exponentiated family of distributions, including the new variant TLExLx, is a notable advancement in distribution theory. This innovative distribution reflects the continuous research efforts aimed at enhancing existing models and expanding the toolkit of statistical methods.

The Type I Half-logistic exponentiated (TIHLEt) distribution has CDF and PDF given respectively as:

$$F(x;\alpha,\lambda,\varpi)=\frac{1-\left[1-\left[G(x;\varpi)\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[G(x;\varpi)\right]^{\alpha}\right]^{\lambda}} \tag{3}$$

$$f(x;\alpha,\lambda,\varpi)=\frac{2\lambda\alpha g(x;\varpi)\left[G(x;\varpi)\right]^{\alpha-1}\left[1-\left[G(x;\varpi)\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[G(x;\varpi)\right]^{\alpha}\right]^{\lambda}\right]^{2}} \tag{4}$$

where ϖ is the parameter vector of the baseline distribution, $G(x;\varpi)$ is the CDF and $g(x;\varpi)$ is the PDF of the baseline distribution with parameter vector ϖ.

The objective of this study is to enhance the flexibility and robustness of the classical Lomax distribution by incorporating a family of distributions and introducing two additional shape parameters (α and λ).

2. Derivation of Type I Half-logistic Exponentiated Lomax (TLExLx) Distribution

In this section, we introduce a novel continuous probability distribution called the Type I Half-logistic exponentiated Lomax (TLExLx) distribution. To assess its shape, we illustrate the distribution's characteristics by plotting its PDF and hazard rate function (HRF). Furthermore, CDF of the TLExLx distribution is obtained by substituting equation (1) into equation (3), leading to the expression:

$$F(x;\alpha,\lambda,\theta,\beta)=\frac{1-\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}} \tag{5}$$

By substituting equations (1) and (2) into equation (4), the PDF of the TLExLx distribution is obtained and it is given as:

$$f(x;\lambda,\alpha,\theta,\beta)=\frac{2\lambda\alpha\theta\beta(1+\beta x)^{-(\theta+1)}\left[1-(1+\beta x)^{-\theta}\right]^{\alpha-1}\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^{2}} \tag{6}$$

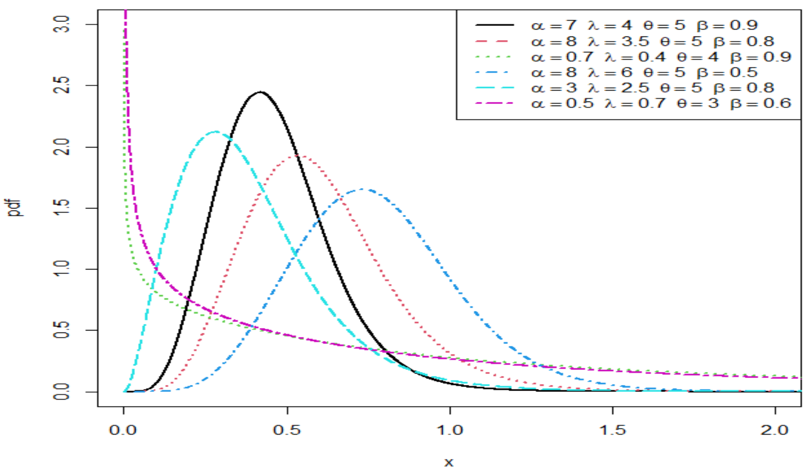

Figure 1: Plots of the PDF of TLExLx distribution showing the shape of the distribution with different parameter values.

The PDF plot of the TLExLx distribution in Figure 1 indicates positive skewness, with the distribution's tail concentrated on the right-hand side.

3. Important Representation

In this section, equation (5) and equation (6) are expanded in linear form and the results are used to obtain some of the statistical and mathematical properties of the TLExLx distribution.
By employing binomial expansion on the denominator of equation (6), we have

$$\left[1+\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^{-2}=\sum_{i=0}^{\infty}(-1)^{i}\binom{1+i}{i}\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda i}$$

$$\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda(i+1)-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\lambda(i+1)-1}{j}\left[1-(1+\beta x)^{-\theta}\right]^{\alpha j}$$

$$\left[1-(1+\beta x)^{-\theta}\right]^{\alpha(j+1)-1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{\alpha(j+1)-1}{k}(1+\beta x)^{-\theta}$$

After mathematical computation, we arrived at

$$f(x;\lambda,\alpha,\theta,\beta)=2\lambda\alpha\theta\beta\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1+i}{i}\binom{\lambda(1+i)-1}{j}\binom{\alpha(j+1)-1}{k}(1+\beta x)^{-\theta(1+k)+1} \quad (7)$$

Equation (7) is the linear representation of equation (6). Also, using equation (5),

$$[F(x;\lambda,\alpha,\theta,\beta)]^h=\left[1-\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^h\left[1+\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^{-h}$$

$$\left[1-\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^h=\sum_{m=0}^{h}(-1)^m\binom{h}{m}\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda m}$$

$$\left[1+\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^{-h}=\sum_{p=0}^{h}(-1)^p\binom{h+p-1}{p}\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda p}$$

$$\left[1-\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda(p+m)}=\sum_{z}^{\infty}(-1)^z\binom{\lambda(p+m)}{z}\left[1-(1+\beta x)^{-\theta}\right]^{\alpha z}$$

$$\left[1-(1+\beta x)^{-\theta}\right]^{\alpha z}=\sum_{q=0}^{\infty}(-1)^q\binom{\alpha z}{q}(1+\beta x)^{-\theta q}$$

$$[F(x;\lambda,\alpha,\theta,\beta)]^h=\sum_{m,p=0}^{h}\sum_{q=0}^{\infty}(-1)^{m+p+z+q}\binom{h}{m}\binom{h+p-1}{p}\binom{\lambda(m+p)}{z}\binom{\alpha z}{q}(1+\beta x)^{-\theta q} \quad (8)$$

Equation (8) is the linear representation of equation (5).

4. Properties of Type I Half-logistic exponentiated Lomax (TLExLx) Distribution

4.1 Moment

$$E(X^r)=\int_0^{\infty}x^r f(x)dx \quad (9)$$

By using the important representation of the PDF in equation (7), we have

$$2\alpha\lambda\theta\beta\int_0^{\infty}x^r(1+\beta x)^{-\theta(k+1)+1}dx$$

$$y=1+\beta x\Rightarrow x=\frac{y-1}{\beta},dx=\frac{dy}{\beta}$$

$$2\alpha\lambda\theta\beta\int_{\infty}^{0}\left(\frac{y-1}{\beta}\right)^r y^{-\theta(k+1)+1}\frac{dy}{\beta}$$

$$2\alpha\lambda\frac{\theta}{\beta^r}\int_0^{\infty}(y-1)^r y^{-\theta(k+1)+1}dy$$

where $\int_0^{\infty}(y-1)^r y^{-\theta(k+1)+1}dy=B[r+1,\theta(i+1)-r]$.

Therefore,

$$E(X^r)=2\lambda\alpha\psi\left(\frac{\theta}{\beta^r}\right)B\left[r+1,\theta(i+1)-r\right], \quad (10)$$

where $\psi = \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{1+i}{i} \binom{\lambda(1+i)-1}{j} \binom{\alpha(j+1)-1}{k}$. When *r=1* in equation (10), the mean of TLExLx distribution is obtained.

4.2 Moment generating function

Moment generating function of a distribution is obtained as

$$M_{(x)}(t) = \int_0^{\infty} e^{tx} f(x) dx$$

since the series expansion for e^{tx} is given as

$$e^{tx} = \sum_{w=0}^{\infty} \frac{(tx)^w}{w!}$$

Using the method of moments, we obtain the moment-generating function of the TLExLx distribution as:

$$M_{(x)}(t) = 2\lambda\alpha\zeta \left(\frac{\theta(i+1)}{\beta^w} \right) B\left[w+1, \theta(i+1) - w \right], \tag{11}$$

where $\zeta = \sum_{w=0}^{\infty} \frac{t^w}{w!} \sum_{i,j,k=0}^{\infty} (-1)^{i,j,k} \binom{1+i}{i} \binom{\lambda(1+i)-1}{j} \binom{\alpha(j+1)-1}{k}$.

4.3 Reliability function

The reliability function is defined as

$$S(x) = 1 - F(x) \tag{12}$$

Subsequently, the TLExLx distribution's survival function is expressed as

$$S(x) = \frac{2\left[1 - \left[1 - \left[1 + \beta(x)\right]^{-\theta}\right]^{\alpha}\right]^{\lambda}}{1 + \left[1 - \left[1 - \left[1 + \beta(x)\right]^{-\theta}\right]^{\alpha}\right]^{\lambda}} \tag{13}$$

4.4 Hazard Function

The expression for the hazard function is as follows

$$h(x) = \frac{f(x)}{S(x)} \tag{14}$$

Subsequently, the hazard function of TILEtLx distribution is expressed as

$$h(x) = \frac{\lambda\alpha\theta\beta\left[1 + \beta(x)\right]^{-(\theta+1)}\left[1 - \left[1 + \beta(x)\right]^{-\theta}\right]^{\alpha-1}}{\left[1 + \left[1 - \left[1 - \left[1 + \beta(x)\right]^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]\left[1 - \left[1 - \left[1 + \beta(x)\right]^{-\theta}\right]^{\alpha}\right]}. \tag{15}$$

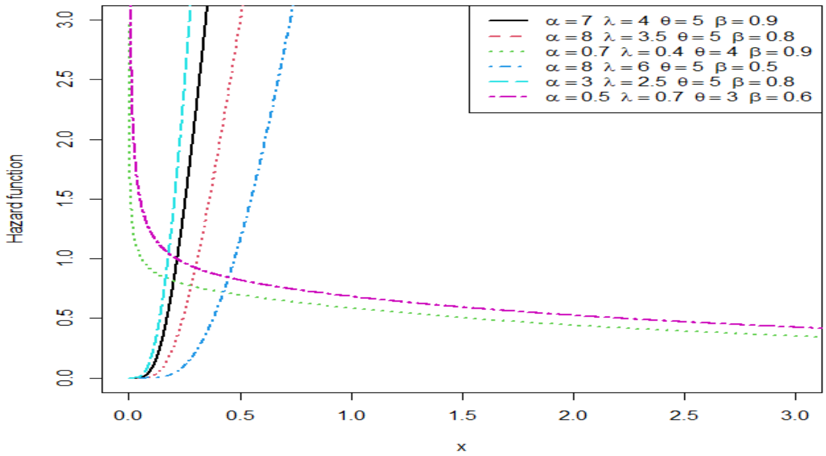

Figure 2: Hazard function plot of the TLExLx distribution showing the shape of the distribution with different parameter value.

The Hazard function plot of the TLEX distribution exhibits a monotonically increasing and decreasing hazard shape.

4.5 Quantile Function

Quantile function is an important function in generating random numbers from any probability distribution. It is the inverse of CDF. Quantile function is obtained using

$$Q(u) = F^{-1}(u) \quad . \tag{16}$$

The quantile function of TLExLx distribution is given as

$$x = Q(u) = \frac{1}{\beta}\left[\left[1-\left[1-\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{-\theta}} - 1\right].$$

4.6 Order Statistics

If $X \sim$ TLExLx distribution, then the pdf of the r^{th} order statistics of $X_{r:n}$ is expressed as

$$f_{r:n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \tag{17}$$

The PDF of r^{th} order statistic for TLExLx distribution is obtained by replacing h with $v + r - 1$ in equation (8). Thus we have

$$f_{r:n}(x) = 2\lambda\alpha\theta\beta \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} \sum_{m,p=0}^{v+r-1} \sum_{i,j,k=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{m+p+z+q+v+i+j+k} \binom{1+i}{i} \binom{\lambda(1+i)-1}{j} \binom{\alpha(j+1)-1}{k} \binom{n-r}{v} \binom{v+r-1}{m} \binom{v+r+p-2}{p} \binom{\lambda(m+p)}{z} \binom{\alpha z}{q} (1+\beta x)^{-\theta(1+k+q)+1} \tag{18}$$

The pdf of minimum order statistic of the TLExLx distribution is obtained by setting $r = 1$ in equation (18)

$$f_{1:n}(x) = 2\lambda\alpha\theta\beta n \sum_{v=0}^{n-1}\sum_{m,p=0}^{v}\sum_{i,j,k=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{m+p+z+q+v+i+j+k}\binom{1+i}{i}\binom{\lambda(1+i)-1}{j}\binom{\alpha(j+1)-1}{k} \binom{n-1}{v}\binom{v+}{m}\binom{v+p-1}{p}\binom{\lambda(m+p)}{z}\binom{\alpha z}{q}(1+\beta x)^{-\theta(1+k+q)+1} \quad (19)$$

Also, the pdf of maximum order statistic of the TLExLx distribution is derived by setting $r = n$ in equation (18)

$$f_{n:n}(x) = 2\lambda\alpha\theta\beta n \sum_{m,p=0}^{v+n-1}\sum_{i,j,k=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{m+p+z+q+v+i+j+k}\binom{1+i}{i}\binom{\lambda(1+i)-1}{j}\binom{\alpha(j+1)-1}{k}\binom{v+n-1}{m} \binom{v+n+p-2}{p}\binom{\lambda(m+p)}{z}\binom{\alpha z}{q}(1+\beta x)^{-\theta(1+k+q)+1} \quad (20)$$

5. Maximum Likelihood Estimation

In this section, we utilize the maximum likelihood estimation (MLE) technique to determine the parameter values of the TLExLx distribution. Through MLE, we aim to find the most probable values for these parameters, enabling us to effectively characterize and understand the behavior of the TLExLx distribution based on the available data. For a random sample, $X_1, X_2, ..., X_n \sim TLExLx(\alpha,\beta,\theta,\lambda)$, the log-likelihood function $L(\alpha,\beta,\theta,\lambda)$ of (6) is given as

$$\begin{aligned}\log L = & n\log(2) + n\log(\lambda) + n\log(\alpha) + n\log(\theta) + n\log(\beta) \\ & -(\theta+1)\sum_{i=1}^{n}\log(1+\beta x_i) + (\alpha-1)\sum_{i=1}^{n}\log\left[1-(1+\beta x_i)^{-\theta}\right] \\ & +(\lambda-1)\sum_{i=1}^{n}\log\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right] \\ & -2\sum_{i=1}^{n}\log\left[1+\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]\end{aligned} \quad (21)$$

Differentiating equation (21) with respect to each parameter and equating to 0, we have

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right] - 2\sum_{i=1}^{n}\frac{\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda}\log\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}}{1+\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda}} = 0 \quad (22)$$

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} = & \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1-(1+\beta x_i)^{-\theta}\right] - (\lambda-1)\sum_{i=1}^{n}\frac{\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\log\left[1-(1+\beta x_i)^{-\theta}\right]}{1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}} \\ & +2\lambda\sum_{i=1}^{n}\frac{\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda-1}\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\log\left[1-(1+\beta x_i)^{-\theta}\right]}{1+\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda}} = 0\end{aligned} \quad (23)$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \log(1+\beta x_i) + (\alpha-1)\sum_{i=1}^{n} \frac{(1+\beta x_i)^{-\theta}\log(1+\beta x_i)}{1-(1+\beta x_i)^{-\theta}}$$
$$+\alpha(\lambda+1)\sum_{i=1}^{n} \frac{\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha-1}(1+\beta x_i)^{-\theta}\log(1+\beta x_i)}{1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}} \tag{24}$$
$$-2\lambda\alpha\sum_{i=1}^{n} \frac{\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda-1}\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha-1}(1+\beta x_i)^{-\theta}\log(1+\beta x_i)}{1+\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda}} = 0$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - (\theta+1)\sum_{i=1}^{n} \frac{\beta x_i x_i}{1+\beta x_i} + (\alpha-1)\theta$$
$$\sum_{i=1}^{n} \frac{(1+\beta x_i)^{-\theta-1}\beta x_i x_i}{(1+\beta x_i)^{-\theta}} + (\lambda-1)\theta\alpha\sum_{i=1}^{n} \frac{\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha-1}(1+\beta x_i)^{-\theta-1}\beta x_i x_i}{1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}} \tag{25}$$
$$-2\lambda\alpha\theta\sum_{i=1}^{n} \frac{\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda-1}\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha-1}(1+\beta x_i)^{-\theta-1}\beta x_i x_i}{1+\left[1-\left[1-(1+\beta x_i)^{-\theta}\right]^{\alpha}\right]^{\lambda}} = 0$$

The ML estimates of the parameters λ, α, θ and β cannot easily be solved analytically due to the nonlinear system of equations existing in an unclosed form. Therefore, in this study, R statistical software is utilized to solve them numerically.

6. Applications to Real-life Data Sets

In this section, we utilize the TLExLx distribution to evaluate the adaptability and robustness of the novel model on three separate real-world datasets. The comparative study involves the examination of various established distributions, including Topp Leone Lomax (TLLx), exponentiated Lomax (EtLx), and Lomax (Lx) distributions. To assess how well these distributions fit the datasets, we employ two commonly used statistical measures: the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC). Using these criteria, we can objectively gauge the performance of each distribution and pinpoint the one that produces the lowest AIC and BIC values, signifying the most suitable match for each specific dataset. The inclusion of these evaluation metrics enables us to gain valuable understanding regarding the efficacy of the TLExLx distribution in comparison to other well-established distributions for modeling real-life datasets. This statistical analysis will shed light on how effectively the TLExLx distribution captures data patterns and how it stacks up against existing distributions in terms of providing a plausible parametric fit to the observed data.

The first dataset originates from Hinkley (1977) and consists of thirty consecutive observations of March precipitation in Minneapolis/St. Paul, measured in inches. The data is presented in Table 1 as follows:

Table 1: First Data Set

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05

The second dataset is obtained from Murthy et al. (2004) and pertains to the time intervals between failures for repairable items. The data is presented in Table 2 as follows:

Table 2: Second Data Set

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17

The third dataset encompasses the total sum of skin folds in 202 athletes, which was collected at the Australian Institute of Sports and referenced by Hosseini et al. (2018). The data set is given in Table 3 as follows:

Table 3: Third Data set

28.0, 98, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2,101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6,52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48, 61.2, 171.1, 113.5, 148.9, 49.9, 59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2, 70.0, 97.2, 123.6, 181.7, 136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9

Table 4: The ML estimates and fit of the models based on data set 1

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	**AIC**	**BIC**
TLExLx	4.745	3.487	0.330	47.227	37.108	82.216	86.521
TLLx	-	0.420	6.035	2.394	38.765	83.529	87.733
EtLx	-	0.163	4.767	9.401	38.384	82.768	86.972
Lx	-	0.008	-	73.863	45.488	94.976	97.778

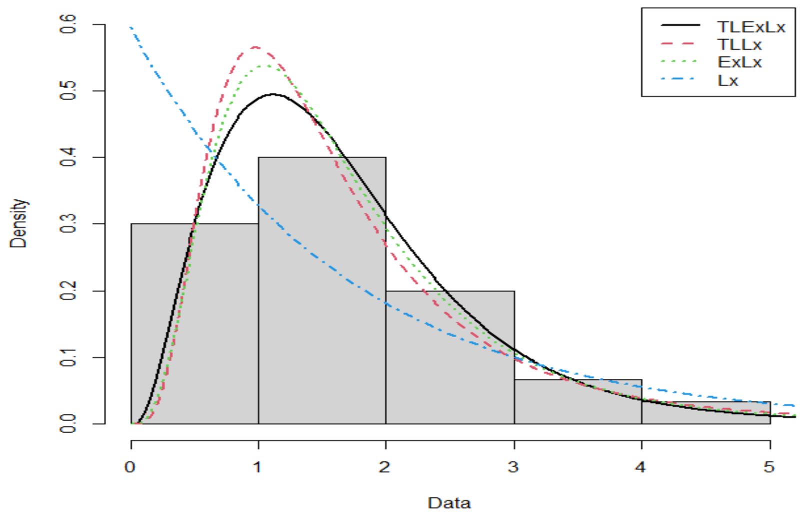

Figure 3: Histogram and fitted models for data set 1

Table 5: The ML estimates and fit of the models derived from data set 2

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	AIC	BIC
TLExLx	5.887	10.940	0.268	49.531	32.002	72.005	76.717
TLLx	-	0.164	3.131	4.134	33.242	72.484	76.816
EtLx	-	0.216	3.275	6.631	33.258	72.516	76.848
Lx	-	0.003	-	241.928	35.763	75.526	77.882

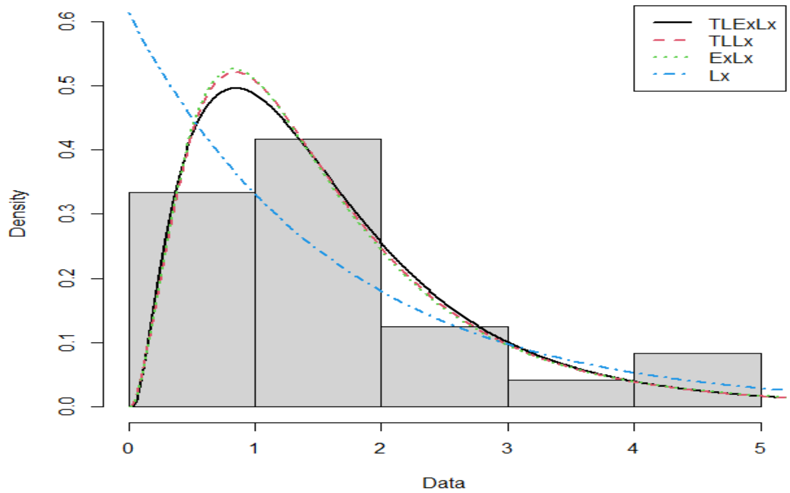

Figure 4: Histogram and fitted models for data set 2

Table 6: The ML estimates and fit of the models based on data set 3

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	AIC	BIC
TLExLx	38.783	0.008	12.800	0.620	954.405	1916.809	1930.042
TLLx	-	0.001	9.838	15.378	957.127	1920.253	1930.178
EtLx	-	0.005	13.311	10.873	955.920	1917.840	1930.765
Lx	-	0.004	-	3.604	1078.429	2160.855	2167.475

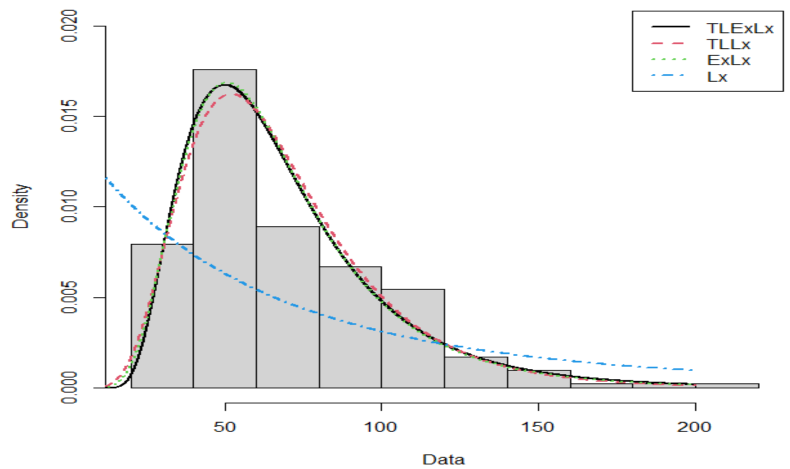

Figure 5: Histogram and fitted models for data set 3

It is observed that the TLExLx model consistently yields the lowest goodness-of-fit scores, as indicated by the negative log-likelihood, AIC, and BIC metrics, across all three datasets, as shown in Tables 4 to 6. This suggests that the TLExLx distribution performs better than the other competing distributions in terms of fitting the data. Furthermore, the histogram plots displayed in Figures 3 to 5 provide additional support for this conclusion, indicating that the TLExLx distribution exhibits greater flexibility compared to its competitors.

7. Conclusion

This study introduces the Type I Half-logistic exponentiated Lomax distribution, a novel model with enhanced flexibility due to its incorporation of four parameters. Our results clearly demonstrate the distribution's effectiveness in representing diverse datasets. Beyond proposing the distribution, we establish a set of statistical and mathematical properties for it, providing essential tools for researchers. We also explore order statistics for this distribution, enriching our understanding of its behavior. The parameters of the distribution were estimated using the maximum likelihood method of estimation. To assess its performance, we conduct tests on real-life datasets, consistently showcasing its superiority over comparison models. Density graphs visually highlight its excellent fit to the data. The Type I Half-logistic exponentiated Lomax distribution is a significant advancement in distribution theory, making it applicable to complex datasets. Its performance in real-world scenarios and comprehensive statistical properties validate its practicality and theoretical importance. This work paves the way for further applications, promising exciting developments in statistical modeling across diverse fields.

8. References

Bello, O.A., Doguwa, S.I., Yahaya, A., and Jibril, H. M. (2021a). A type I Half Logistic Exponentiated-G family of distrbutions: Properties and Application. *Communication in Physical Sciences,* 7(3), 147-163

Bello, O. A., Doguwa, S. I., Yahaya, A. and Jibril, H. M. (2021b). A Type II Half-Logistic Exponentiated-G Family of Distributions with Applications to Survival Analysis. *FUDMA Journal of Sciences*, 5(3): 177-190.

Elgarhy, M., Haq M. A., Ozel, G. and Nasir, M. A. (2017). A New Exponentiated Extended Family of Distributions with Applications. *Gazi University Journal of Science, 30(2): 101-115.*

Elgarhy, M., Nasir, M. A., Jamal, F. and Ozel, G. (2018). The type II Topp-Leone generated family of distributions : Properties and applications. *Journal of Statistics and Management Systems*, 21(8):1529-1551

Cordeiro, G.M. and de Castro, M. (2011) A New Family of Generalized Distributions. *Journal of Statistical Computation and Simulation*, 81, 883-898.

Falgore, J. Y. and Doguwa, S. I. (2020). Kumaraswamy-Odd Rayleigh-G Family of Distributions with Applications. *Open Journal of Statistics*, 10, 719-734.

Hassan, A.S. and Abd-Allah, M. (2018). Exponentiated Weibull–Lomax distribution: Properties and estimation. *Journal of Data Science*., 16, 277–298.

Hassan, A. S and Al-Ghamdi, A. S (2009). Optimum step stress accelerated life testing for Lomax distribution. *Journal of Applied Scientific Research*, 5, 2153-2164.

Holland, O., Golaup, A. and Aghavami, A. H. (2006). Traffic characteristics of aggregated module downloads for mobile terminal reconfiguration. *IEE Proceedings on Communications*, 135, 683-690.

Hinkley, D. (1977). On quick choice of power transformations, *Journal of the Royal Statistical Series C*, 26(1), 67-69.

Hosseini, B., Afshari, M. and Alizadeh M. (2018). The generalized odd gamma-G family of distributions: properties and applications. *Austrian Journal of Statistics*, 47(2):69-89

Ibrahim, S., Doguwa, S. I., Audu, I. and Jibril, H. M. (2020a). On the Topp Leone exponentiated-G Family of Distributions: Properties and Applications, *Asian Journal of Probability and Statistics*, 7(1): 1-15.

Ibrahim, S., Doguwa, S. I., Isah, A. and Haruna, J. M. (2020b). The Topp Leone Kumaraswamy-G Family of Distributions with Applications to Cancer Disease Data. *Journal of Biostatistics and Epidemiology*, 6(1) : 37-48.

Ijaz, M, Asim, S.M. and Alamgir (2019). Lomax exponential distribution with an application to real life data. *PLoS ONE*, 14(12):e0225827.

Lomax, K.S. (1954). Business failures: Another example of the analysis of failure data, *Journal of the American Statistical Association*, 49, 847-852.

Mead, M.E. (2016). On five-parameter Lomax distribution: Properties and applications. *Pakistan Journal of Statistics and Operation Research*, 12, 185–200.

Murthy, D.N.P., Xie, M. and Jiang, R. (2004). Weibull Models, Wiley Series in Probability and Statistics, John Wiley and Sons, New York.

Nagarjuna, B.V., Vishnu Vardhan, R. and Chesneau, C. (2021a). Kumaraswamy Generalized Power Lomax Distribution and Its Applications. *Stats*, 4, 28–45.

Nagarjuna, B.V. Vishnu Vardhan, R. and Chesneau, C. (2021b). On the Accuracy of the Sine Power Lomax Model for Data Fitting. *Modelling*, 2, 78–104.

Nagarjuna, B.V. Vishnu Vardhan, R. and Chesneau, C. (2022). Nadarajah–Haghighi Lomax Distribution and Its Applications. *Mathematics and Computational Applications*, 27, 30.

Rady, E.H.A., Hassanein, W.A. and Elhaddad, T.A. (2016). The power Lomax distribution with an application to bladder cancer data. *SpringerPlus*, 5, 1–22.

Sanusi, A. A., Doguwa, S. I. S., Audu, I. and Baraya, Y. M. (2020). Topp Leone Exponential – G Family of Distributions: Properties and Application, *NIPES Journal of Science and Technology Research*, 2(4) : 83-96.

Sule, I., Doguwa, S.I., Isah, A. and Jibril, H.M. (2021a). Some properties and applications of Topp Leone Kumaraswamy Lomax distribution, *Journal of Statistical Modeling and Analytics*, 3(2) : 81-94.

Sule, I., Doguwa, S.I., Isah, A. and Jibril, H.M. (2021b).Topp Leone exponentiated Lomax distribution and its applications to breast and bladder cancer data, *Journal of Probability and Statistical Science*, 19(1) : 67-81.

Tahir, M.H., Cordeiro, G.M., Mansoor, M. and Zubair, M. (2015). The Weibull-Lomax distribution: properties and applications, *Hacettepe Journal of Mathematics and Statistics*, 44(2): 455-474